Asteroseismic Diagnostics of Misaligned Cores and Envelopes

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We examine the asteroseismic phenomenology of stellar rotation when the **direction of the rotational axis** varies with radial position.

Motivation

In the **shellular approximation** of rotation, the angular momenta of all mass shells are **assumed to be aligned.** However, this may not necessarily be true, as, e.g.:

- **Tidal realignment** of oblique planets exerts **torque** on CZ (e.g. Nick Saunders's poster)
- **Engulfment** deposits angular momentum **directly** into stellar envelope

Both result in stratification of the axis of rotation!

- What are the **observational signatures** of this misalignment?
- How will these affect our **existing rotational measurements**?

Generalised rotational splitting matrices

Neglecting latitudinal differential rotation, conventional rotational multiplet components are the **eigensystem of a linear operator** with matrix elements

$$\left\langle \boldsymbol{\xi}_{\ell m}, \hat{\mathcal{R}}\boldsymbol{\xi}_{\ell m'} \right\rangle = \beta_{n\ell} \int \mathbf{J}_z \Omega(r) K_{n\ell}(r)\, \mathrm{d}r, \quad \mathbf{J}_z = \mathrm{diag}\left(-\ell, \ell+1, \ldots, \ell-1, \ell\right), \tag{1}$$

in the basis of spherical harmonics.

$\mathbf{J}_z$ is one of three basis matrices in the ℓth representation of $\mathfrak{so}(3)$ — i.e. of rank $2\ell+1$ — along with $\mathbf{J}_x$ and $\mathbf{J}_y$. These satisfy commutation relations

$$[\mathbf{J}_i, \mathbf{J}_j] = -i\epsilon_{ijk}\mathbf{J}_k. \tag{2}$$

If at each radial position r we have not only $\Omega(r)$ but in general $\boldsymbol{\Omega}(r) = \Omega(r)\hat{\mathbf{n}}(r)$, then we now need to find the eigensystem of

$$\mathbf{R}_{nl} = \beta_{n\ell} \int K_{n\ell}(r) \left[\boldsymbol{\Omega}(r) \cdot \vec{\mathbf{J}}\right] \mathrm{d}r. \tag{3}$$

Eigenvalues of this matrix are **generalised multiplet splittings**, and eigenvectors — forming some unitary matrix $\mathbf{U} \in \rho_\ell[\mathrm{SO}(3)]$ — specify a change of basis (in spherical harmonics) from the reference coordinate system, to those of the normal modes. In particular, $\hat{\mathbf{n}} \cdot \vec{\mathbf{J}} = \mathbf{d}^\ell \mathbf{J}_z \mathbf{d}^{\ell,\dagger}$, where $\mathbf{d}^\ell$ is Wigner's d-matrix: $\implies \delta\omega_m \sim m\delta\omega_{m=1}$ **even with misalignment**.

Two-Zone Model of Misalignment

Consider a rotating core (with constant Ω_{core}) aligned with the z-axis, and an envelope (constant Ω_{env}) aligned with unit vector $\hat{\mathbf{n}}$; the boundary between the two sits at radius r_0. We define a relative sensitivity parameter α for a given multiplet, and misalignment angle β, as

$$\alpha_{n\ell} = \int_0^{r_0} K_{n\ell}(r)\, \mathrm{d}r; \quad \cos\beta = \mathbf{e}_z \cdot \hat{\mathbf{n}}. \tag{4}$$

Effective rotational splitting is then

$$\delta\omega_{\text{rot},n\ell} \sim m\beta_{n\ell}|\boldsymbol{\Omega}_{\text{eff}}|, \tag{5}$$

where $\boldsymbol{\Omega}_{\text{eff}}$ is found from **vector addition** (fig. 1). By the triangle inequality, we have that

$$|\boldsymbol{\Omega}_{\text{eff}}| \leq \alpha|\boldsymbol{\Omega}_{\text{core}}| + (1-\alpha)|\boldsymbol{\Omega}_{\text{env}}|. \tag{6}$$

- Misalignment **systematically reduces** rotational splittings compared to the aligned case!
- Amount depends on **both** misalignment angle β and sensitivity α (fig. 2).

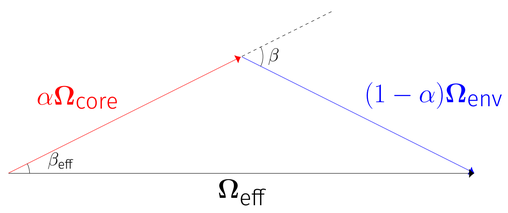

Figure 1: Geometric picture of misaligned rotational splittings as addition of angular momentum vectors

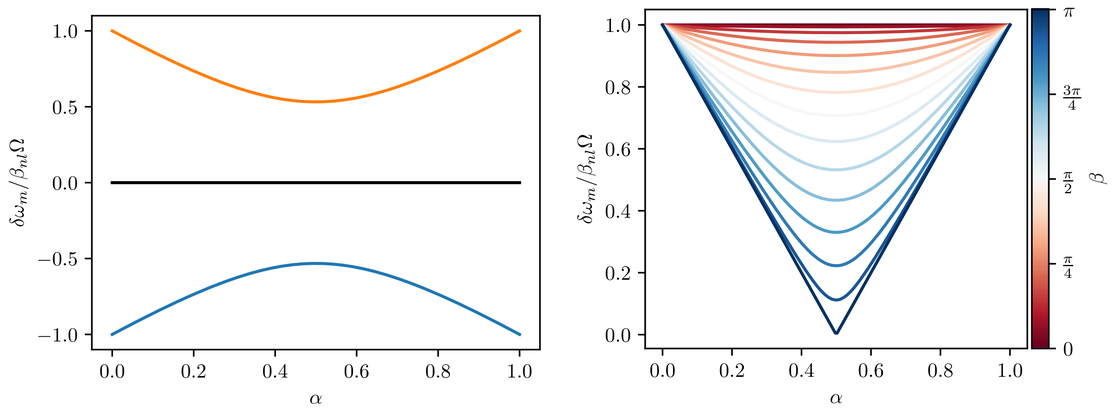

Figure 2: Suppression of rotational splitting in presence of misalignment. **Left**: Schematic representation of rotational splitting when $\Omega_{\text{core}} = \Omega_{\text{env}}$ with $\cos\beta = 0$, showing dependence on α. Intermediate values of α between 0 and 1 reduce the splitting width. **Right**: The amount of reduction depends on the value of β.

The **apparent inclination** implied by the relative amplitude of multiplet components will **also depend on** α **and** β (fig. 3).

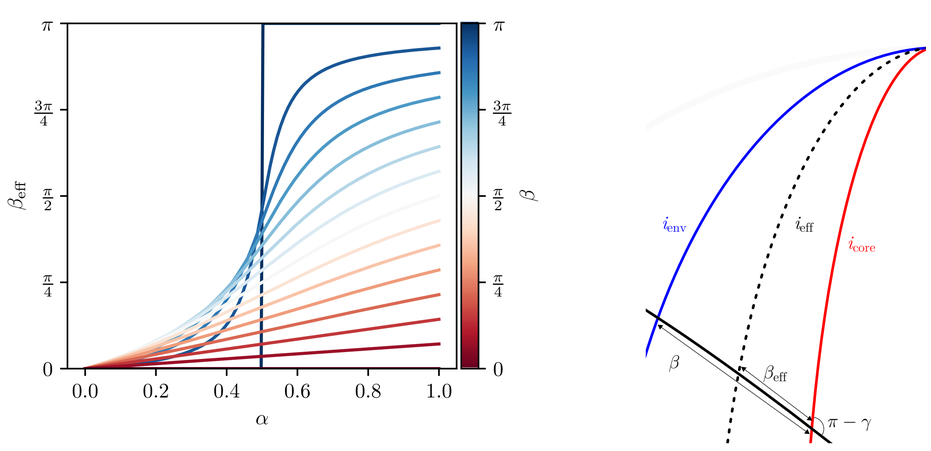

Figure 3: Dependence of the apparent inclination angle on α and β. **Left**: Numerical dependence on α and β of an effective β (as in fig. 1) associated with the multiplet. **Right**: Relationship between i_{core}, i_{env}, and i_{eff}, on a spherical triangle. Note the geometrical degeneracy owing to an underspecified projection angle γ.

Mode Coupling in the Two-Zone Model

If these zones coincide with propagation regions of a **mixed-mode solar-like oscillator**, rotation interacts with mode coupling via a **Quadratic Hermitian Eigenvalue Problem**. The unitary matrices $\mathbf{d}^\ell$ modify this coupling as

$$\left(\begin{bmatrix} -\boldsymbol{\Omega}_p^2 & \mathbf{A} \\ \mathbf{A}^\dagger & -\boldsymbol{\Omega}_g^2 \end{bmatrix} \otimes \mathbb{I}_{2\ell+1} + \omega \begin{bmatrix} \mathbf{R}_p \otimes \mathbf{d}^\ell \mathbf{J}_z \mathbf{d}^{\ell,\dagger} & 0 \\ 0 & \mathbf{R}_g \otimes \mathbf{J}_z \end{bmatrix} + \omega^2 \begin{bmatrix} \mathbb{I}_p & \mathbf{D} \\ \mathbf{D}^\dagger & \mathbb{I}_g \end{bmatrix} \otimes \mathbb{I}_{2\ell+1}\right) \mathbf{c} = 0, \tag{7}$$

where $\otimes$ is the tensor (Kronecker) product.

- Upon misalignment, mixed modes become linear combinations of p- and g-modes with different m_p and m_g.
- Since $\alpha = \frac{\zeta}{2-\zeta}$, **different mixing fractions** ζ will yield **different apparent inclinations**!
- Suggests **reinterpretation of existing inclination measurements** made using mixed modes (e.g. Kepler-56: Huber et al., 2013)

Near-degeneracy Effects: Modified Avoided Crossings

Modes of $m_p \neq m_g$ may couple to each other, depending on the amount of misalignment. The morphology of rotational avoided crossings is likewise modified (fig. 4).

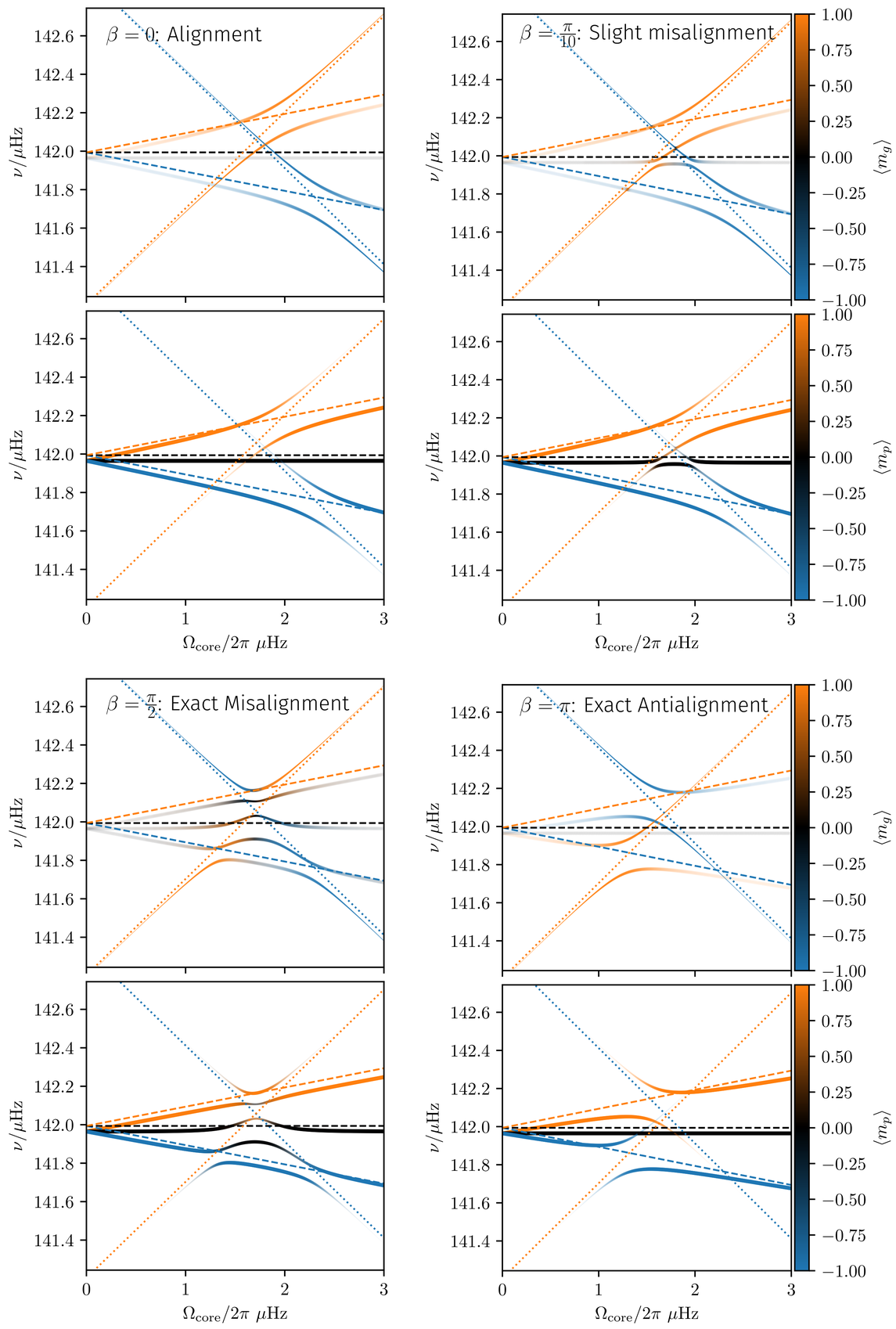

Figure 4: Rotational avoided crossings in the presence of rotational misalignment, computed with respect to a red giant MESA model (Model I of Ong et al., 2022). Each set of two panels shows the same set of avoided crossings, coloured variously by either m_p or m_g, for a single value of the core-envelope misalignment angle β.

Key Takeaways

- Stratified misalignment **systematically reduces** rotational splittings, **modifies the shape** of rotational avoided crossings, and **may be diagnosed** by per-multiplet variability of the implied inclination angle. However, inferences of the misalignment angle are **geometrically degenerate** without further constraints from e.g. Rossiter-McLaughlin measurements.

References

Huber, D., Carter, J. A., Barbieri, M., et al. 2013, Science, 342, 331

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